# Oh the Morphisms!

Connecting Seemingly Unconnected Morphisms

Alan J. Ponte
University of California, Berkeley
alanjponte@berkeley.edu

**Abstract**

The goal of this paper is to develop a simple way of classifying different morphisms. For many Mathematics undergraduates, there is a plethora of morphisms to memorize, and the task can be quite daunting. This task becomes especially cumbersome when one must know a large amount of properties in order to prove simple results. If this paper is successful, the reader will have a firm understanding of certain morphisms. However, a larger benefit may be from developing a new way of thinking about morphisms. This paper is aimed toward an undergraduate Mathematics student, but the concepts should be applicable to most with mathematical maturity.

**Contents:**



## 0. Introduction and Preliminaries

In abstract mathematics, it is often nice to be able to describe objects, and describe actions on the objects. You may be already familiar with the notion of a function, which is a way of describing actions between two sets of objects.

**Definition 0.1.1** (function). A **function** between two sets $A$ and $B$ is a mapping, denoted $f: A \to B; x \longmapsto f(x)$.

Where we say that $A$ is the domain of the function, and $B$ is the codomain of the function. Also, the function is almost useless unless we know what the function does to elements of A. In this case, we denote the output of the function as $f(x)$.

Now if we think of morphisms as functions, we can see that it is really just a mapping between objects. So a ring homomorphism is a mapping between rings, a group homomorphism is a mapping between groups, etc.

Furthermore, we have nice properties of functions which when applied, make working with functions a lot less painful.

**Definition 0.1.2**. Let $f: A \to B$ be a function.

- A function is **injective** if and only if $\forall x, y \in A,\ if\ f(x) = f(y)$, then $x = y$.
- A function is **surjective** if and only if $\forall b \in B,\ \exists a \in A$ such that $f(a) = b$.
- A function is **bijective** if f is both injective and surjective.

Before taking higher math classes, these concepts are quite difficult to grasp. However, these properties show up *everywhere* in morphisms.

## 1. The Linear Morphism

The first morphism that most students might see is a linear morphism, which is just a glorified map between two vector spaces. Recall, that a vector space is just a collection of objects which we call vectors, and we have certain operations that we can do to these vectors. For example, we can add vectors, scale vectors, etc.

Of course we can't talk about vectors without talking about number fields. Recall, a nonempty set $\mathbb{K}$ is a **Number Field** if

(i) $\mathbb{Z} \subseteq \mathbb{K}$

(ii) There are well-defined notions of addition, subtraction, multiplication, and division that all obey the usual notions of arithmetic.

Recall that $\mathbb{Z}$ is not a number field, because there is no notion of inverses.

Now that we've cleared up a few details, we can begin talking about a linear morphism.

**Definition 1.1**. Let *V, W* be $\mathbb{K} - vector\ spaces$.

- A function

$$f: V \to W; v \mapsto f(v),$$

is called a **Linear Morphism** if :

(i) $\forall u, v \in V$, we have $f(u+v) = f(u) + f(v)$, where '+' on the LHS is '+' in $V$ and '+' on the RHS is '+' in $W$.

(ii) $\forall u \in V,\ \lambda \in \mathbb{K}$, we have $f(\lambda v) = \lambda f(v)$. Where the scalar multiplication on the LHS is in $V$, and scalar multiplication on the RHS is in $W$.

We can already see the sort of things that morphisms do. A morphism is a function, so it's nice to have already a good grasp on the notion of a function. In fact, we can already utilize some of the properties of functions; we use the terms injective and surjective to describe linear morphisms that satisfy the same conditions as a surjectivity and injectivity of a function. Also, a bijective linear morphism is called an isomorphism[1]. The notion of an isomorphism is very important in algebra.

[1] If two sets, $A$, and $B$, are isomorphic, we tend to write $A \cong B$.

A very useful property, which arises all the time in morphisms, is the notion of a kernel. Without being pedantic, recall that the kernel of a linear morphism is essentially "everything in $V$ which maps to $0$ in $W$. That is, given the morphism $f: V \to W$, $kerf = \{v \in V \mid f(v) = 0_W\}$

We further abstract out this notion of a linear morphism by created a basis of a vector space. Given two vector spaces which are isomorphic, and utilizing a basis, we can essentially perform the same operations between the two vector spaces, and the mapping will preserve the operation. The notions of getting two 'sets', which are essentially the same, and creating a mapping between them, will show up in our other morphism.

## 2. The Group Homomorphism

Now that we've knocked out the boring morphism, let's move on to the ones which many students of abstract mathematics will see; the collection of homomorphisms. But first, we must define what the structures are that we are operating on.

**Definition 2.1** A **Group**, denoted $(G, +, *)$, is a set $G$ combined with two binary operations $+, *$, in which:

(i) There exists a neutral element $e$ such that $\forall g \in G, ge = eg = g$.

(ii) There exists an inverse $g^{-1}$, such that $gg^{-1} = g^{-1}g = e$.

(iii) '+' is associative. That is, $\forall a, b, c \in G, a + (b + c) = a + (b + c)$.

Note that we are not talking about commutivity. That is a special class of Groups, called abelian.

Now, what do you expect should be the next thing we do between these groups? That's right, we will create a mapping between them.

**Definition 2.2** Let G,H, be groups. A function $\phi: G \to H$ is a **Group Homomorphism** if:

(i) $\phi(1_g) = 1_H$

(ii) $\forall g \in G, h \in H, \phi(g * h) = \phi(g) * \phi(h)$.

Just as with linear morphisms, we have a kernel for group homomorphisms. The kernel of a group homomorphism $\phi\ G \to H$ is denoted $ker\phi = \{g \in G \mid \phi(g) = 1_H\}$.

And again, a group homomorphism is said to be injective and surjective if it follows the same properties of surjectivity and injectivity of functions. If a group homomorphism is bijective, then it is an Isomorphism.

Again, we can see how nice it is to work in an isomorphism. We could have to completely disjoint groups, but if we are allowed to create an isomorphism between them, we will essentially be operating on the same group.

Recall, that with given a Vector Space $V$, and a basis $B \subseteq V$, we can find *any* linear combination of vectors in $B$. In groups we have a similar structure; we call this a subgroup.

**Definition 2.3** Let $H, G$ be groups. $H \subseteq G$ is a **subgroup** of $G$ if it is a group with respect to the binary operation associated with $G$.

This definition is a bit useless right now, so let's look at a concrete example. For any group $G, x \in G$, let $< x >$ be the subset defined by $\{x^k \middle| k \in \mathbb{Z}\}$. The proof that this is a subgroup is trivial, but shall be left to the reader. This in fact is the *cyclic subgroup generated by x*. So for any $0 \neq x \in (\mathbb{R}, *)$, say $\sqrt{2}, < \sqrt{2} > =$

$\left\{\cdots,\frac{1}{\sqrt{2}},\frac{1}{2},1,\sqrt{2},2,\cdots\right\}$. We can see that we are essentially just taking $\mathbb{Z}-linear$ combinations of $\sqrt{2}$. If this reminds you of a basis vector, it's because they are quite similar. And of course this leads to a nice result, that given an arbitrary two groups, we can create a mapping between them, by either utilizing and isomorphism, or by showing that each group can be generated by the same subgroup.

An important result of subgroups is that given a group $G$, and subgroup $H \subseteq G$, we *always* have the canonical map $\phi: G \to G\backslash H$ ; $g \longmapsto \grave{g}$, where $\grave{g}$ arises from an equivalence relation defined on $G/H$[2].

## 2.2 The Isomorphism Theorems for Groups

Now that we've accessed the necessary components of a homomorphism, let us describe some very interesting results. These results are usually referred to as "the isomorphism theorems for groups".

**Theorem 2.2.1** Let $G$ be a group, $H \subseteq G$ subgroup such that $\phi: G \to H$. Then $G \;\cong G\backslash ker\phi$.

# 3. The Ring Homomorphism

Another homomorphism, which is quite prevalent in abstract mathematics, is a ring homomorphism. But to first achieve the task of describing a ring homomorphism, we must have a concrete description of a ring.

**Definition 3.1** A ring is a set with a binary operation $*$, denoted $(R,*)$, such that:

(i) '$*$' is associtative.
(ii) There exists a neutral element. That is, $\forall r \in R, \exists e \text{ such that } r*e = e*r = r$ .

Again, we have a set that we can perform operations on. So the natural transition is to create a function between rings.

**Definition 3.2** Let $R, S$ be rings. A function $\psi: R \to S$ is a **Ring Homomorphism** satisfying:

(i) $\forall a,b \in R, \psi(a+b) = \psi(a) + \psi(b)$.
(ii) $\forall a,b \in R, \psi(ab) = \psi(a)\psi(b)$.

What good is a morphism without our good friend the kernel? Yes, even ring homomorphisms have kernels, denoted $ker\psi = \{r \in R \mid \psi(r) = 0_S\}$.

Recall, that given a basis of a vector space $V$, we could obtain every linear combination of vectors in $V$. We have a similar concept in Rings; called an ideal.

**Definition 2.3** Let $I \subseteq R$ be a subset of a ring R. I is an **Ideal** of R if:

(1) $I \neq \emptyset$.

[2] If the reader is not familiar with equivalence relations, they are urged to look at a few examples.

(2) $\forall r, s \in I\ (r + s) \in I$.
(3) $\forall r \in R, i \in I, (ri) \in I$.
Alternatively, we could say that $I$ forms all linear combinations of elements in $I \subset R$. Notationally, we write $I = < r_1, r_2, \cdots, r_k >$, for some $r_i \in R$, and is read "I is generated by elements $r_1, r_2, \cdots, r_k$". Without bogging down notion further, let us look at Ideals in the ring of integers $\mathbb{Z}$. Can you see that every element of $\mathbb{Z}$ is generated by a single element? Let's take an example of $1 \in \mathbb{Z}$. $I = < 1 > = \{\ 1 + 2, 1 + 2, \cdots, 1 +\ -1, 1 +\ -2, \cdots, 1 +\ k\mathbb{Z}\} \subseteq \mathbb{Z}$, for some $k \in \mathbb{Z}$. The proof is left to the reader.
As we can see, ideals "look like" basis vectors.

## 4. Conclusion

As we have shown, many sets behave similarly. We now have a handy table (4.1), which allows us to show the similarities. The main takeaway from this should be that a morphism is nothing more than a map between sets, with certain operations which are pre-defined within those sets.
Furthermore, there are a plethora of objects in which we could describe morphisms between them. If interested, the reader is encouraged to research Modules, which are essentially vector spaces over any set, not restricting us to a number field. However, many of these morphisms are beyond the scope of this paper.

Figure 4.1

| | **Linear Morphism** $f: V \to W$ | **Group Hom.** $\phi: G \to H$ | **Ring Hom.** $\psi: R \to S$ |
|---|---|---|---|
| 'Type' of object | Vectors | Groups | Rings |
| Kernel | $ker f = \{v \in V \mid f(v) = 0_W\}$ | $ker \phi = \{g \in G \mid \phi(g) = 1_H\}$ | $ker \psi = \{r \in R \mid \psi(r) = 0_s\}$ |
| How to get linear combinations | Bases, $B \in V$ | Subgroups, $H \subseteq G$ | Ideals $I \subseteq R$ |

**Acknowledgments:**
[1] Professor Nathan Ilten
nilten@math.berkeley.edu
[2] George Melvin for his helpful notes on Linear Morphisms
gmelvin@math.berkeley.edu

**References:**

[1] Shilov, Georgi E., *Linear Algebra,* Dover Publications 1977.
[2] Dummit, David S., Foote, Richard M., *Abstract Algebra*, John Wiley & Sons, Inc. 2004.